

# A Class of Periodic Continued Radicals

Costas J. Efthimiou

## Abstract

We compute the limits of a class of periodic continued radicals and we establish a connection between them and the fixed points of the Chebycheff polynomials.

## 1 Introduction.

Continued radicals

$$a_0 \sqrt{b_1 + a_1 \sqrt{b_2 + a_2 \sqrt{b_3 + a_3 \sqrt{b_4 + \cdots}}}}$$

have been well known among mathematicians [1,3-6,9-11] and they have even appeared at mathematical competitions. For example, Ramanujan's famous result [2]

$$\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \cdots}}}} = 3$$

was given as one of the problems on the Putnam Mathematical Competition in 1966. However, the literature on them is rather scant and, with only some exceptions, the main results consider cases with positive numbers  $a_i$  and  $b_i$ . Among the results on continued radicals with negative coefficients are some by Ramanujan himself [2], such as the continued radical

$$\sqrt{a - \sqrt{a + \sqrt{a + \sqrt{a - \sqrt{a + \cdots}}}}}$$

with period 3, and Problem 1174 in [4] (motivated by a 1953 Putnam Problem for which  $a = 7$ )

$$\sqrt{a - \sqrt{a + \sqrt{a - \sqrt{a + \sqrt{a - \cdots}}}}}$$

with period 2.

## 2 The Problem.

In this brief article we find the values for a class of periodic continued radicals of the form

$$a_0 \sqrt{2 + a_1 \sqrt{2 + a_2 \sqrt{2 + a_3 \sqrt{2 + \cdots}}}}, \quad (1)$$

where for some positive integer  $n$ ,

$$a_{n+k} = a_k, \quad k = 0, 1, 2, \dots,$$

and

$$a_k \in \{-1, +1\}, \quad k = 0, 1, \dots, n-1.$$

The product

$$P = \prod_{k=0}^{n-1} a_k$$

will be called the parity of the radical.

Obviously, depending on the choice of the  $a_k$ 's, the pattern may have period less than  $n$ . For example, given any  $n$ , if  $a_k = 1$  for all  $k$ , then the pattern has period 1, giving the well studied radical

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}}$$

It is easy to find a formula for the number of radicals of minimal period  $n$ : Given the radical (1), there are  $2^n$  different ways to choose the periodic pattern. However, some of these patterns will have period 1, some period 2, and so on up to period  $n$  for all periods  $d$  that are divisors of  $n$ . Given  $n$ , let's denote by  $N(d)$  the number of radicals with period  $d$ . Then

$$\sum_{d|n} N(d) = 2^n .$$

As is well known, this equation can be inverted with the help of the Möbius function  $\mu(n)$ :

$$N(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) 2^d .$$

### 3 The Answer.

Towards our result, we present the following lemma [8].

**Lemma 1.** For  $\alpha_i \in \{-1, 1\}$ ,  $i = 0, 1, \dots, n-1$ ,

$$2 \sin \left[ \left( \alpha_0 + \frac{\alpha_0 \alpha_1}{2} + \cdots + \frac{\alpha_0 \alpha_1 \cdots \alpha_{n-1}}{2^{n-1}} \right) \frac{\pi}{4} \right] = \alpha_0 \sqrt{2 + \alpha_1 \sqrt{2 + \alpha_2 \sqrt{2 + \cdots + \alpha_{n-1} \sqrt{2}}}} .$$

The lemma is easily proven by induction.

According to this lemma, the partial sums of the continued radical (1) are given by

$$x_n = 2 \sin \left[ \left( a_0 + \frac{a_0 a_1}{2} + \cdots + \frac{a_0 a_1 \cdots a_{n-1}}{2^{n-1}} \right) \frac{\pi}{4} \right] .$$

The series

$$\alpha_0 + \frac{\alpha_0 \alpha_1}{2} + \cdots + \frac{\alpha_0 \alpha_1 \cdots \alpha_{n-1}}{2^{n-1}} + \cdots$$

is absolutely convergent and thus it converges to some number  $\alpha$ . Therefore the original continued radical converges to the real number

$$x = 2 \sin \frac{\alpha \pi}{4} .$$

Alternatively this can be written

$$x = 2 \cos \frac{\beta \pi}{2} , \quad \beta = 1 - \frac{\alpha}{2} .$$

We can find a concise formula for  $x$ . For this calculation it is more useful to use the products

$$P_m = \prod_{k=0}^m a_k , \quad m = 0, 1, \dots, n-1 .$$

Of course  $P_{n-1} = P$  is the parity of the radical. In this notation,

$$\alpha = P_0 + \frac{P_1}{2} + \frac{P_2}{2^2} + \cdots + \frac{P_{n-1}}{2^{n-1}} + \frac{P_0 P_{n-1}}{2^n} + \cdots ,$$

which we can easily rewrite as

$$\begin{aligned} \alpha &= \left( P_0 + \frac{P_1}{2} + \frac{P_2}{2^2} + \cdots + \frac{P_{n-1}}{2^{n-1}} \right) \left( 1 + \frac{P}{2^n} + \frac{P^2}{2^{2n}} + \cdots \right) \\ &= \left( P_0 + \frac{P_1}{2} + \frac{P_2}{2^2} + \cdots + \frac{P_{n-1}}{2^{n-1}} \right) \frac{2^n}{2^n - P} . \end{aligned}$$

Then

$$\frac{\beta\pi}{2} = 2\ell \frac{\pi}{2^n - P} ,$$

where

$$2\ell = 2^{n-1} - P - (P_{n-2} + 2P_{n-3} + \cdots + 2^{n-3}P_1 + 2^{n-2}P_0) .$$

Despite its simplicity, this result may not be easily interpreted. Some examples might help. In Tables 1 through 3 we give the value  $x$  of the continued radical for all possible choices of the  $a_k$ 's when  $n = 2, 3$  and 4. The answers take an amazingly compact form. Looking at these tables and the previous result, we easily realize that when the parity is even then

$$x = 2 \cos \left( \frac{2\pi\ell}{2^n - 1} \right) , \quad \ell = 0, 1, \dots, 2^{n-1} - 1 ,$$

and when the parity is odd then

$$x = 2 \cos \left( \frac{2\pi\ell}{2^n + 1} \right) , \quad \ell = 1, \dots, 2^{n-1} .$$

$\alpha_0$	$\alpha_1$	$P$	$\frac{x}{2} = \sin \frac{\alpha\pi}{4}$	$\frac{x}{2} = \cos \frac{\beta\pi}{2}$
-1	-1	+1	$\sin \left( -\frac{\pi}{6} \right)$	$\cos \frac{2\pi}{3}$
-1	+1	-1	$\sin \left( -\frac{3\pi}{10} \right)$	$\cos \frac{4\pi}{5}$
+1	-1	-1	$\sin \left( \frac{\pi}{10} \right)$	$\cos \frac{2\pi}{5}$
+1	+1	+1	$\sin \left( \frac{\pi}{2} \right)$	$\cos 0$

Table 1: The value of our continued radical for all choices of the  $a_k$ 's when  $n = 2$ . We see that when they are chosen such that  $\alpha_0\alpha_1 = 1$ , the continued radical equals  $2 \cos \left( \frac{2\pi\ell}{2^2 - 1} \right)$ ,  $\ell = 0, 1$ , and when they are chosen such that  $\alpha_0\alpha_1 = -1$ , the continued radical equals  $2 \cos \left( \frac{2\pi\ell}{2^2 + 1} \right)$ ,  $\ell = 1, 2$ .

Motivated by the special cases, we can prove the result in full generality. For  $m = 0, 1, \dots, n-2$  we define

$$Q_m = \frac{1 + P_m}{2} .$$

Since  $P_m \in \{-1, 1\}$ ,  $Q_m \in \{0, 1\}$ . Inversely,  $P_m = 2Q_m - 1$ . We now evaluate  $2\ell$  in terms of the  $Q_m$ 's:

$$\begin{aligned} 2\ell &= 2^{n-1} - P - [(2Q_{n-2} - 1) + 2(2Q_{n-3} - 1) + \cdots + 2^{n-3}(2Q_1 - 1) + 2^{n-2}(2Q_0 - 1)] \\ &= 2^{n-1} - P + (1 + 2 + \cdots + 2^{n-3} + 2^{n-2}) - 2(Q_{n-2} + 2Q_{n-3} + \cdots + 2^{n-3}Q_1 + 2^{n-2}Q_0) \\ &= 2^{n-1} - P + (2^{n-1} - 1) - 2(Q_{n-2} + 2Q_{n-3} + \cdots + 2^{n-3}Q_1 + 2^{n-2}Q_0) , \end{aligned}$$

or

$$2\ell = 2^n - P - 1 - 2Q,$$

where

$$Q = Q_{n-2} + 2Q_{n-3} + \cdots + 2^{n-3}Q_1 + 2^{n-2}Q_0$$

is the integer whose binary expression is  $\overline{Q_0Q_1\cdots Q_{n-3}Q_{n-2}}$ . Now we notice that when

$\alpha_0$	$\alpha_1$	$\alpha_2$	$P$	$\frac{x}{2} = \sin \frac{\alpha\pi}{4}$	$\frac{x}{2} = \cos \frac{\beta\pi}{2}$
-1	-1	-1	-1	$\sin\left(-\frac{\pi}{6}\right)$	$\cos \frac{6\pi}{9}$
-1	-1	+1	+1	$\sin\left(-\frac{\pi}{14}\right)$	$\cos \frac{4\pi}{7}$
-1	+1	-1	+1	$\sin\left(-\frac{5\pi}{14}\right)$	$\cos \frac{6\pi}{7}$
-1	+1	+1	-1	$\sin\left(-\frac{7\pi}{18}\right)$	$\cos \frac{8\pi}{9}$
+1	-1	-1	+1	$\sin\left(\frac{3\pi}{14}\right)$	$\cos \frac{2\pi}{7}$
+1	-1	+1	-1	$\sin\left(\frac{\pi}{18}\right)$	$\cos \frac{4\pi}{9}$
+1	+1	-1	-1	$\sin\left(\frac{5\pi}{18}\right)$	$\cos \frac{2\pi}{9}$
+1	+1	+1	+1	$\sin\left(\frac{\pi}{2}\right)$	$\cos 0$

Table 2: The value of our continued radical for all choices of the  $a_k$ 's when  $n = 3$ . We see that when they are chosen such that  $\alpha_0\alpha_1\alpha_2 = 1$ , the continued radical equals  $2\cos\left(\frac{2\pi\ell}{2^3-1}\right)$ ,  $\ell = 0, 1, 2, 3$ , and when they are chosen such that  $\alpha_0\alpha_1\alpha_2 = -1$ , the continued radical equals  $2\cos\left(\frac{2\pi\ell}{2^3+1}\right)$ ,  $\ell = 1, 2, 3, 4$ .

$\alpha_0$	$\alpha_1$	$\alpha_2$	$\alpha_3$	$P$	$\frac{x}{2} = \sin \frac{\alpha\pi}{4}$	$\frac{x}{2} = \cos \frac{\beta\pi}{2}$
-1	-1	-1	-1	+1	$\sin\left(-\frac{\pi}{6}\right)$	$\cos \frac{10\pi}{15}$
-1	-1	-1	+1	-1	$\sin\left(-\frac{7\pi}{34}\right)$	$\cos \frac{12\pi}{17}$
-1	-1	+1	-1	-1	$\sin\left(-\frac{3\pi}{34}\right)$	$\cos \frac{10\pi}{17}$
-1	-1	+1	+1	+1	$\sin\left(-\frac{\pi}{30}\right)$	$\cos \frac{8\pi}{15}$
-1	+1	-1	-1	-1	$\sin\left(-\frac{11\pi}{34}\right)$	$\cos \frac{14\pi}{17}$
-1	+1	-1	+1	+1	$\sin\left(-\frac{3\pi}{10}\right)$	$\cos \frac{12\pi}{15}$
-1	+1	+1	-1	+1	$\sin\left(-\frac{13\pi}{30}\right)$	$\cos \frac{14\pi}{15}$
-1	+1	+1	+1	-1	$\sin\left(-\frac{15\pi}{34}\right)$	$\cos \frac{16\pi}{17}$
+1	-1	-1	-1	-1	$\sin\left(\frac{5\pi}{34}\right)$	$\cos \frac{6\pi}{17}$
+1	-1	-1	+1	+1	$\sin\left(\frac{7\pi}{30}\right)$	$\cos \frac{4\pi}{15}$
+1	-1	+1	-1	+1	$\sin\left(\frac{\pi}{10}\right)$	$\cos \frac{6\pi}{15}$
+1	-1	+1	+1	-1	$\sin\left(\frac{\pi}{34}\right)$	$\cos \frac{8\pi}{17}$
+1	+1	-1	-1	+1	$\sin\left(\frac{11\pi}{30}\right)$	$\cos \frac{2\pi}{15}$
+1	+1	-1	+1	-1	$\sin\left(\frac{9\pi}{34}\right)$	$\cos \frac{4\pi}{17}$
+1	+1	+1	-1	-1	$\sin\left(\frac{13\pi}{34}\right)$	$\cos \frac{2\pi}{17}$
+1	+1	+1	+1	+1	$\sin\left(\frac{\pi}{2}\right)$	$\cos 0$

Table 3: The value of our continued radical for all choices of the  $a_k$ 's when  $n = 4$ . We see that when they are chosen such that  $\alpha_0\alpha_1\alpha_2\alpha_3 = 1$ , the continued radical equals  $2\cos\left(\frac{2\pi\ell}{2^4-1}\right)$ ,  $\ell = 0, 1, 2, \dots, 7$ , and when they are chosen such that  $\alpha_0\alpha_1\alpha_2\alpha_3 = -1$ , the continued radical equals  $2\cos\left(\frac{2\pi\ell}{2^4+1}\right)$ ,  $\ell = 1, 2, \dots, 8$ .

we go through all possible sequences  $(a_k)_{k=0}^{n-1}$ , the sequence  $(P_k)_{k=0}^{n-1}$  will go through all possible sequences of  $\pm 1$ 's, and therefore the sequence  $(Q_k)_{k=0}^{n-2}$  will go through all possible sequences of 0's and 1's, with each such sequence appearing once with each value of the parity  $P = P_{n-1} = \pm 1$ . Consequently, the integer  $Q$  will run through the integers from 0 to  $2^{n-1} - 1$  once with each parity. Thus, when  $P = 1$

$$\ell = (2^{n-1} - 1) - Q ,$$

and, as  $Q$  runs through the integers from 0 to  $2^{n-1} - 1$ ,  $\ell$  will run through the same values in reverse order. When  $P = -1$  we get

$$\ell = 2^{n-1} - Q ,$$

which will then run through all the values from 1 to  $2^{n-1}$ .

In the following section, we give an alternative way to look at this result: a nice connection with the Chebycheff polynomials.

## 4 Chebycheff Polynomials.

The  $N$ th Chebycheff polynomial of the first kind is defined by

$$T_N(\cos \theta) = \cos(N\theta) .$$

Now consider the quadratic polynomial  $P(x) = x^2 - 2$  defined on  $[-2, 2]$ . Using the substitution  $x = 2 \cos \theta$ , it is easy to see that  $P(x) = 2 \cos(2\theta)$  and

$$P^n(x) = 2 \cos(2^n \theta) .$$

In other words  $P^n(x) = 2T_{2^n}(x/2)$ . The fixed points of  $P^n(x)$  are given by  $P^n(x) = x$ , or

$$2 \cos(2^n \theta) = 2 \cos \theta .$$

This equation is easily solved to give the  $2^n$  solutions

$$\begin{aligned} \theta &= \frac{2\pi\ell}{2^n - 1} , \quad \ell = 0, 1, \dots, 2^{n-1} - 1 , \\ \theta &= \frac{2\pi\ell}{2^n + 1} , \quad \ell = 1, \dots, 2^{n-1} . \end{aligned}$$

The fixed points are then  $x = 2 \cos \theta$ .

On the other hand, we can find these fixed points as follows. The equation  $P^n(x) = x$  can be written as  $P(P^{n-1}(x)) = x$ . Using the expression of  $P(x)$  we can solve for  $P^{n-1}(x)$ :

$$P^{n-1}(x) = \pm \sqrt{x + 2} .$$

Repeating this  $n$  times we find

$$x = \pm \sqrt{2 \pm \sqrt{2 \pm \sqrt{2 \pm \dots \pm \sqrt{2 + x}}}}$$

This nested radical reproduces our continued radicals if we iteratively replace  $x$  in the right-hand side by this expression.

## 5 Conclusion.

We have proved that the radicals given by equation (1) have limits two times the fixed points of the Chebycheff polynomials  $T_{2^n}(x)$ , thus unveiling an interesting relation between these topics.

In [11], the authors defined the set  $S_2$  of all continued radicals of the form (1) (with  $a_0 = 1$ ) and they investigated some of its properties by assuming that the limit of the radicals exists. With this note, we have partially bridged this gap. It is straightforward to see that the limit exists, but we have identified it only for periodic radicals.

The continued radical

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}} = 2$$

is well known, while

$$\sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 - \cdots}}}} = 2 \sin \frac{\pi}{18}$$

is a special case of Ramanujan's radical (appearing explicitly in [2]).

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*Department of Physics, University of Central Florida, Orlando, FL 32816*  
*costas@physics.ucf.edu*